Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT
JOURNAL OF Algebra
Journal of Algebra 267 (2003) 156–177
www.elsevier.com/locate/jalgebra

Pseudo Cohen–Macaulay and pseudo generalized Cohen–Macaulay modules ☆

Nguyen Tu Cuong [a,*] and Le Thanh Nhan [b]

[a] *Institute of Mathematics, PO Box 631, Boho, 10.000 Hanoi, Viet Nam*
[b] *Thai Nguyen Pedagogical University, Viet Nam*

Received 16 April 2002

Communicated by Craig Huneke

Abstract

In this paper we study the structure of two classes of modules called pseudo Cohen–Macaulay and pseudo generalized Cohen–Macaulay modules. We also give a characterization for these modules in term of the Cohen–Macaulayness and generalized Cohen–Macaulayness. Then we apply this result to prove a cohomological characterization for sequentially Cohen–Macaulay and sequentially generalized Cohen–Macaulay modules.

Keywords: Local cohomology; Multiplicity; Generalized fractions; Noetherian; Artinian

1. Introduction

Let $(R,\mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $\dim M = d$. For a system of parameters $\underline{x} = (x_1,\dots,x_d)$ of M and a set of positive integers $\underline{n} = (n_1,\dots,n_d)$, we set $\underline{x}(\underline{n}) = (x_1^{n_1},\dots,x_d^{n_d})$. Consider the differences

$$\begin{aligned} I_{M,\underline{x}}(\underline{n}) &= \ell\bigl(M/\underline{x}(\underline{n})M\bigr) - n_1\dots n_d\, e(\underline{x};M),\\ J_{M,\underline{x}}(\underline{n}) &= n_1\dots n_d\, e(\underline{x};M) - \ell\bigl(M/Q_M\bigl(\underline{x}(\underline{n})\bigr)\bigr) \end{aligned}$$

☆ This work is supported in part by the National Basis Research Programme in Natural Science of Vietnam.
* Corresponding author.
E-mail addresses: cuongnt@hn.vnn.vn (N.T. Cuong), trtrnhan@yahoo.com (L.T. Nhan).

0021-8693/03/$ – see front matter
doi:10.1016/S0021-8693(03)00225-4

as functions in $n_1, \ldots, n_d$, where $e(\underline{x}; M)$ is the multiplicity of M with respect to $\underline{x}$ and

$$Q_M(\underline{x}) = \bigcup_{t>0}\left(\left(x_1^{t+1}, \ldots, x_d^{t+1}\right)M : x_1^t \ldots x_d^t\right).$$

It was proved in [CK] that $\ell(M/Q_M(\underline{x}(\underline{n})))$ is just the length of generalized fraction $1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1)$ defined by Sharp and Hamieh [SH]. Therefore, in general, $I_{M,\underline{x}}(\underline{n})$ and $J_{M,\underline{x}}(\underline{n})$ are not polynomials for $n_1, \ldots, n_d$ large enough (see [GK,CMN]), but it is still nice since they are bounded above by polynomials. Especially, the least degree of all polynomials in $\underline{n}$ bounding above $I_{M,\underline{x}}(\underline{n})$ (respectively $J_{M,\underline{x}}(\underline{n})$) is independent of the choice of $\underline{x}$, and it is denoted by $p(M)$ (respectively $pf(M)$). The invariant $p(M)$ is called *the polynomial type* of M (see [C2,CM]). If we stipulate the degree of the zero polynomial is $-\infty$, then M is a Cohen–Macaulay module if and only if $p(M) = -\infty$, and M is a generalized Cohen–Macaulay module if and only if $p(M) \leqslant 0$. Recall that generalized Cohen–Macaulay modules had been introduced in [CST]. In that paper they showed that M is generalized Cohen–Macaulay if and only if $\ell(H_{\mathfrak{m}}^i(M)) < \infty$ for all $i = 1, \ldots, d-1$, where $H_{\mathfrak{m}}^i(M)$ is the ith local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$. However, little is known about structure of M when $p(M) > 0$.

The purpose of this paper is to study modules M which satisfy $pf(M) = -\infty$ or $pf(M) \leqslant 0$. Note that if M is Cohen–Macaulay then $pf(M) = -\infty$, and if M is generalized Cohen–Macaulay then $pf(M) \leqslant 0$. However, the converse is not true. There are many modules M with $pf(M) = -\infty$, but $p(M)$ is large optionally. We will show that if M is of $pf(M) = -\infty$ or $pf(M) \leqslant 0$ then the properties of M are still good and closely related to that of Cohen–Macaulay modules or generalized Cohen–Macaulay modules. Since such modules M are, so to speak, *pseudo Cohen–Macaulay* and *pseudo generalized Cohen–Macaulay*, respectively, it seems interesting to clarify such given modules.

The paper is divided into five sections. In Section 2, we first describe some basic properties of pseudo Cohen–Macaulay modules and pseudo generalized Cohen–Macaulay modules. In particular, it follows that a finite direct sum of pseudo Cohen–Macaulay (respectively pseudo generalized Cohen–Macaulay modules) is pseudo Cohen–Macaulay (respectively pseudo generalized Cohen–Macaulay). In the next section, we give a characterization of these modules as follows.

Theorem. *Let R be a Noetherian local ring admitting a dualizing complex. Let $0 = \bigcap N_i$, where N_i is $\mathfrak{p}_i$-primary, be a reduced primary decomposition of the submodule* 0 *of M. Set*

$$N = \bigcap_{\dim R/\mathfrak{p}_j = d} N_j.$$

Then the following statements are true.

(i) *M is pseudo Cohen–Macaulay if and only if M/N is a Cohen–Macaulay module.*
(ii) *M is pseudo generalized Cohen–Macaulay if and only if M/N is a generalized Cohen–Macaulay module.*

This result will be shown in Theorem 3.1. As corollaries of the theorem, we give properties of pseudo Cohen–Macaulay and pseudo generalized Cohen–Macaulay modules passing to reducing parameter element, relating to the monomial property, with respect to the localization, ... The concept of *sequentially Cohen–Macaulay module* was introduced by Stanley [St] for graded modules. In this paper, by the same way, we introduce this notion for the local case. It follows that the class of sequentially Cohen–Macaulay modules is strictly contained in the class of pseudo Cohen–Macaulay modules. Therefore in Section 4, we are interested in properties of sequentially CM modules. We also introduce the notion of *sequentially generalized Cohen–Macaulay modules* as an extension of the concept of sequentially Cohen–Macaulay modules. Note that the class of pseudo generalized Cohen–Macaulay modules also contain strictly all sequentially generalized Cohen–Macaulay modules. The main result of Section 5 is to give a cohomological characterization of sequentially Cohen–Macaulay modules and sequentially generalized Cohen–Macaulay modules. This characterization will be shown in Theorems 5.1 and 5.3. The notion of *module of deficiency* was studied in [Sch2]. We will show in Proposition 5.6 the unmixedness (unmixedness up to $\mathfrak{m}$-primary component) of the $p(M)$-th module of deficiency of pseudo Cohen–Macaulay (pseudo generalized Cohen–Macaulay) modules over local rings admitting a dualizing complex.

2. Pseudo Cohen–Macaulay and pseudo generalized Cohen–Macaulay modules

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $\dim M = d$. Let $\underline{x} = (x_1, \dots, x_d)$ be a system of parameters of M and $\underline{n} = (n_1, \dots, n_d)$ a set of positive integers. Set

$$I_{M,\underline{x}}(\underline{n}) = \ell\big(M/\big(x_1^{n_1}, \dots, x_d^{n_d}\big)M\big) - n_1 \dots n_d\, e(\underline{x}; M),$$
$$J_{M,\underline{x}}(\underline{n}) = n_1 \dots n_d\, e(\underline{x}; M) - \ell\big(M/Q_M\big(\underline{x}(\underline{n})\big)\big),$$

where $\underline{x}(\underline{n}) = (x_1^{n_1}, \dots, x_d^{n_d})$ and

$$Q_M(\underline{x}) = \bigcup_{t>0}\big(\big(x_1^{t+1}, \dots, x_d^{t+1}\big)M : x_1^{n} \cdots x_d^{t}\big).$$

We consider $I_{M,\underline{x}}(\underline{n})$ and $J_{M,\underline{x}}(\underline{n})$ as functions in $\underline{n}$. It has been proved in [CK] that $\ell(M/Q_M(\underline{x}(\underline{n})))$ is just the length of generalized fraction $1/(x_1^{n_1}, \dots, x_d^{n_d}, 1)$ defined by Sharp and Hamieh [SH]. Therefore, in general, both $I_{M,\underline{x}}(\underline{n})$ and $J_{M,\underline{x}}(\underline{n})$ are not polynomials for $\underline{n}$ large enough (see [GK,CMN]), but these functions always take non-negative values (see [C2,CM]) and bounded above by polynomials. Moreover, we have the following important property.

Theorem 2.1 [C2,CM]. *The following statements are true.*

(i) *The least degree of all polynomials in $\underline{n}$ bounding above the function $I_{M,\underline{x}}(\underline{n})$ is independent of the choice of $\underline{x}$.*

(ii) *The least degree of all polynomials in* $\underline{n}$ *bounding above the function* $J_{M,\underline{x}}(\underline{n})$ *is independent of the choice of* $\underline{x}$.

The least degree in Theorem 2.1(i) is called *polynomial type* of M and denoted by $p(M)$. The least degree in Theorem 2.1(ii) is denoted by $pf(M)$.

Definition 2.2. (i) M is called a *pseudo Cohen–Macaulay* module (pseudo CM module for short) if $pf(M) = -\infty$.

(ii) M is called a *pseudo generalized Cohen–Macaulay* module (pseudo generalized CM module for short) if $pf(M) \leqslant 0$.

We denote by $\widehat{R}$ the $\mathfrak{m}$-adic completion of R and $\widehat{M}$ the $\mathfrak{m}$-adic completion of M. Then we have by [CM, 3.4] that $pf(M) = pf(\widehat{M})$. Therefore the pseudo Cohen–Macaulayness and pseudo generalized Cohen–Macaulayness are preserved by $\mathfrak{m}$-adic completion.

Lemma 2.3. *The following statements are true.*

(i) M *is pseudo CM if and only if so is* $\widehat{M}$.
(ii) M *is pseudo generalized CM if and only if so is* $\widehat{M}$.

Lemma 2.4. *Let* N *be a submodule of* M *such that* $\dim N < d$. *Then we have*

(i) M *is pseudo CM if and only if* M/N *is pseudo CM.*
(ii) M *is pseudo generalized CM if and only if* M/N *is pseudo generalized CM.*

Proof. Since $\dim N < d$, we have $\operatorname{Ann} N \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$ with $\dim A/\mathfrak{p} = d$. Thus there exists a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ with $x_1 \in \operatorname{Ann} N$. Put $\overline{M} = M/N$. Then it is easy to check that $\overline{M}/Q_{\overline{M}}(\underline{x}(\underline{n})) \cong M/Q_M(\underline{x}(\underline{n}))$ for all sets of positive integers $\underline{n} = (n_1, \ldots, n_d)$. Therefore $J_{M,\underline{x}}(\underline{n}) = J_{\overline{M},\underline{x}}(\underline{n})$. Thus $pf(M) = pf(\overline{M})$ and the lemma follows from the Definition 2.2. $\square$

Lemma 2.5. *The following statements are true.*

(i) *A direct sum of finitely many pseudo CM modules is pseudo CM.*
(ii) *A direct sum of finitely many pseudo generalized CM modules is pseudo generalized CM.*

Proof. (i) It is enough to prove for a direct sum of two modules. Let $M = M_1 \oplus M_2$, where M_1 and M_2 are pseudo CM. The case of $\dim M_1 \neq \dim M_2$ follows easily from Lemma 2.4. Suppose that $\dim M_1 = \dim M_2$. Let $\underline{x}$ be a system of parameters of M. Then $\underline{x}$ is also a system of parameters of M_1 and M_2. For any set of positive integers $\underline{n} = (n_1, \ldots, n_d)$, it is clear that $e(\underline{x}(\underline{n}); M) = e(\underline{x}(\underline{n}); M_1) + e(\underline{x}(\underline{n}); M_2)$. Moreover, it is easily to check that

$$Q_M\big(\underline{x}(\underline{n})\big) = Q_{M_1}\big(\underline{x}(\underline{n})\big) \oplus Q_{M_2}\big(\underline{x}(\underline{n})\big).$$

Therefore $J_{M,\underline{x}}(\underline{n}) = J_{M_1,\underline{x}}(\underline{n}) + J_{M_2,\underline{x}}(\underline{n})$. Since M_1 and M_2 are pseudo CM, we have $J_{M_1,\underline{x}}(\underline{n}) = 0$ and $J_{M_2,\underline{x}}(\underline{n}) = 0$, for all $\underline{n}$. Hence $J_{M,\underline{x}}(\underline{n}) = 0$, for all $\underline{n}$. Thus M is pseudo CM.

(ii) Follows similarly by the proof of (i). □

The following result of [CM, 3.6] gives us some vanishing (respectively finitely generated) properties of local cohomology modules for pseudo CM (respectively pseudo generalized CM) modules.

Lemma 2.6. *The following statements are true.*

(i) *If M is pseudo CM then $H^i_{\mathfrak{m}}(M) = 0$, for all $i = p(M) + 1, \dots, d - 1$.*
(ii) *If M is pseudo generalized CM then $\ell(H^i_{\mathfrak{m}}(M)) < \infty$, for all $i = p(M)+1, \dots, d-1$.*

3. A characterization of pseudo CM and pseudo generalized CM modules

The following characterization of pseudo CM modules and pseudo generalized CM modules is the main result of this section.

Theorem 3.1. *Let R be a Noetherian local ring admitting a dualizing complex and M a finitely generated R-module. Let $0 = \bigcap N_i$, where N_i is $\mathfrak{p}_i$-primary, be a reduced primary decomposition of the submodule 0 of M. Set*

$$N = \bigcap_{\dim R/\mathfrak{p}_j = d} N_j.$$

Then the following statements are true.

(i) *M is pseudo CM if and only if M/N is a Cohen–Macaulay module.*
(ii) *M is pseudo generalized CM if and only if M/N is a generalized Cohen–Macaulay module. Moreover, in this case*

$$J_{M,\underline{x}}(\underline{n}) = J_{M/N,\underline{x}}(\underline{n}) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell\bigl(H^i_{\mathfrak{m}}(M/N)\bigr)$$

for all systems of parameters $\underline{x}$ and $\underline{n} \gg 0$.

Proof. (ii) Suppose that M is pseudo generalized CM. Since $\dim N < d$, we have by Lemma 2.4(ii) that M/N is pseudo generalized CM. Therefore $\ell(H^i_{\mathfrak{m}}(M/N)) < \infty$ for all $i = p(M/N) + 1, \dots, d - 1$ by Lemma 2.6(ii). Set $\mathfrak{a}_i = \operatorname{Ann}(H^i_{\mathfrak{m}}(M/N))$ for $i = 1, \dots, d - 1$, $\mathfrak{a} = \mathfrak{a}_1 \dots \mathfrak{a}_{d-1}$, and $p = p(M/N)$. We need to show that $p \leqslant 0$. Suppose that $p > 0$. Then we obtain by [C2, 3.1] and [Sch1, 2.4.6] that

$$p = \dim R/\mathfrak{a} = \dim R/\mathfrak{a}_p.$$

On the other hand, it is clear that M/N is equidimensional, i.e. $\dim R/\mathfrak{p} = \dim M/N$, for all minimal prime ideals $\mathfrak{p} \in \operatorname{Supp} M/N$. Moreover, for all $\mathfrak{p} \in \operatorname{Supp} M/N$, we have

$$\operatorname{depth}_{R_\mathfrak{p}}(M/N)_\mathfrak{p} \geqslant \min\{\dim_{R_\mathfrak{p}}(M/N)_\mathfrak{p}, 1\}.$$

So, M/N satisfies Serre's condition (S_1). Therefore $\dim R/\mathfrak{a}_p \leqslant p-1$ by [Sch1, 3.2.1]. It gives a contradiction. Thus $p \leqslant 0$, i.e. M/N is generalized Cohen–Macaulay. Conversely, suppose that M/N is generalized Cohen–Macaulay. Then M/N is pseudo generalized CM. Because $\dim N < d$, we have by Lemma 2.4(ii) that M is pseudo generalized CM. Since M/N is generalized Cohen–Macaulay, the formula follows by [SH, 3.7].

(i) The case where $d \leqslant 1$ is trivial. Let $d > 1$. Suppose that M is pseudo CM. Then M/N is generalized Cohen–Macaulay by (ii). Therefore, for any system of parameters $\underline{x}$ of M/N, we get

$$J_{M/N,\underline{x}}(\underline{n}) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell\big(H^i_\mathfrak{m}(M/N)\big)$$

for all $n \gg 0$. Since M/N is pseudo CM, $J_{M/N,\underline{x}}(\underline{n}) = 0$ for all $\underline{n}$. So, $H^i_\mathfrak{m}(M/N) = 0$ for all $i = 1, \ldots, d-1$. Moreover, it is clear that $H^0_\mathfrak{m}(M/N) = 0$. Thus, M/N is Cohen–Macaulay. Conversely, suppose that M/N is Cohen–Macaulay. Then M/N is pseudo CM. Because $\dim N < d$, we have by Lemma 2.4(i) that M is pseudo CM. □

Remark 3.2. (i) The submodule N of M defined in Theorem 3.1 is exactly the largest submodule of dimension strictly less than d of M (see Lemma 4.4(i) for more details).

(ii) Theorem 3.1 is not true if R does not possess a dualizing complex. For example, let R be the 2-dimension local domain considered by Nagata [N, Appendix, Example 2] (see also [FR]). It follows by [Sch2, 6.1] that $\widehat{R}/I$ is Cohen–Macaulay $\widehat{R}$-module, where I is the largest submodule of $\widehat{R}$ of dimension at most 1. Therefore $\widehat{R}$ is pseudo CM by Lemma 2.4(i) and hence so is R by Lemma 2.3(i). Since $\operatorname{Ass} R = \{0\}$, the largest submodule of R of dimension at most 1 is the zero ideal. But it is well known that R is not a Cohen–Macaulay ring.

Since the complete ring $\widehat{R}$ always admits a dualizing complex, the following result is an immediate consequence of Theorem 3.1.

Corollary 3.3. *Let $0 = \bigcap \widehat{N}_i$, where $\widehat{N}_i$ is $\hat{\mathfrak{p}}_i$-primary, be a reduced primary decomposition of the submodule 0 of $\widehat{R}$-module $\widehat{M}$. Let*

$$\widehat{N} = \bigcap_{\dim \widehat{R}/\hat{\mathfrak{p}}_j = d} \widehat{N}_i.$$

Then the following statements are true.

(i) *M is pseudo CM if and only if $\widehat{M}/\widehat{N}$ is a Cohen–Macaulay $\widehat{R}$-module.*

(ii) *M is pseudo generalized CM if and only if $\widehat{M}/\widehat{N}$ is a generalized Cohen–Macaulay $\widehat{R}$-module.*

The notion of reducing parameter element was introduced by Auslander–Buchsbaum [AB]: a parameter element x of M is called *reducing* if $x \notin \mathfrak{p}$, for all $\mathfrak{p} \in \operatorname{Ass} M$ with $\dim R/\mathfrak{p} \geqslant d-1$. Note that if M is generalized Cohen–Macaulay then every parameter element of M is reducing. Moreover, if M is Cohen–Macaulay then so is M/xM and if M is generalized Cohen–Macaulay then so is M/xM for all parameter element x of M. In the case of pseudo CM or pseudo generalized CM modules, this property still hold for all reducing parameter elements.

Corollary 3.4. *Let x be a reducing parameter element of M. Then we have*

(i) *If M is pseudo CM then so is M/xM.*
(ii) *If M is pseudo generalized CM then so is M/xM.*

Proof. Let $\widehat{N}$ be the largest submodule of $\widehat{M}$ of dimension at most $d-1$. Let $\overline{M} = \widehat{M}/\widehat{N}$. Then we have:

$$\overline{M}/x\overline{M} \cong \frac{\widehat{M}}{\widehat{N} + x\widehat{M}} \cong \frac{\widehat{M}/x\widehat{M}}{(\widehat{N} + x\widehat{M})/x\widehat{M}}. \tag{$*$}$$

Since x is also a reducing parameter element of $\widehat{M}$, it follows that x is $\overline{M}$-regular and if $\dim N = d-1$ then x is a parameter element of $\widehat{N}$. It implies that $\widehat{N} \cap x\widehat{M} = x\widehat{N}$ and $\dim \widehat{N}/x\widehat{N} = d-2$. Therefore we have

$$\dim\bigl(\widehat{N} + x\widehat{M}\bigr)/x\widehat{M} = \dim \widehat{N}/\bigl(\widehat{N} \cap x\widehat{M}\bigr) = \dim \widehat{N}/x\widehat{N} = d-2.$$

On the other hand, it is clear that $\dim(\widehat{N} + x\widehat{M})/x\widehat{M} \leqslant d-2$, if $\dim \widehat{N} \leqslant d-2$. Thus in any case we have

$$\dim\bigl(\widehat{N} + x\widehat{M}\bigr)\big/x\widehat{M} \leqslant d-2. \tag{$**$}$$

Now we prove (i). Suppose that M is pseudo CM. Then $\widehat{M}$ is pseudo CM by Lemma 2.3(i). Therefore $\overline{M}$ is Cohen–Macaulay by Theorem 3.1(i). It implies that $\overline{M}/x\overline{M}$ is Cohen–Macaulay. Therefore we have by $(*)$ that

$$\frac{\widehat{M}/x\widehat{M}}{(\widehat{N} + x\widehat{M})/x\widehat{M}}$$

is Cohen–Macaulay and hence it is pseudo CM. Since $\dim((\widehat{N} + x\widehat{M})/x\widehat{M}) \leqslant d-2$ by $(**)$, it follows by Lemma 2.4(i) that $\widehat{M}/x\widehat{M}$ is pseudo CM and hence so is M/xM by Lemma 2.3(i).

(ii) Follows similarly to the proof of (i). $\square$

A system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M is said to have *the monomial property* if

$$x_1^t \cdots x_d^t M \nsubseteq \left(x_1^{t+1}, \ldots, x_d^{t+1}\right)M$$

for all $t > 0$. Note that $\underline{x}$ has the monomial property if and only if $\ell(M/Q_M(\underline{x})) \neq 0$. Therefore if M is pseudo CM then the monomial property holds for all system of parameters of M. In [H], Hochster has conjectured that any system of parameters of R has the monomial property. He also showed that in general the monomial property does not hold for modules, but it holds for high powers of system of parameters, i.e. there exists for each system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M a positive integer $n(\underline{x})$, which in general depends on $\underline{x}$, such that

$$\left(x_1^{n_1}\right)^t \cdots \left(x_d^{n_d}\right)^t M \nsubseteq \left(\left(x_1^{n_1}\right)^{t+1}, \ldots, \left(x_d^{n_d}\right)^{t+1}\right)M$$

for all $n_1, \ldots, n_d \geqslant n(\underline{x})$ and $t \geqslant 0$. Therefore it seems to be interesting to find a concrete uniform bound for such number $n(\underline{x})$. The following result is to solve this problem for pseudo generalized CM modules. Let $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal of R. A system of parameters. $(x_1, \ldots, x_d)$ of M is said to be a *weak* $\mathfrak{q}$*-sequence* if

$$(x_1, \ldots, x_{i-1})M :_M x_i \subseteq (x_1, \ldots, x_{i-1})M :_M \mathfrak{q} \quad \text{for } i = 1, \ldots, d.$$

It was proved in [SV] that if M is generalized Cohen–Macaulay then there exists an $\mathfrak{m}$-primary ideal q such that every system of parameters of M is a weak $\mathfrak{q}$-sequence.

Corollary 3.5. *Suppose that M is pseudo generalized CM. Let $\widehat{N}$ be the largest submodule of $\widehat{M}$ of dimension at most $d-1$ and $\underline{x} = (x_1, \ldots, x_d)$ a system of parameters of M. Then we have $\widehat{M}/\widehat{N}$ is generalized Cohen–Macaulay. Let $\mathfrak{q}$ be a $\mathfrak{m}$-primary ideal such that every system of parameters of $\widehat{M}/\widehat{N}$ is a weak $\mathfrak{q}$-sequence. If $x_i \in \mathfrak{m}\mathfrak{q}$ for some i, then $\underline{x}$ has the monomial property. In particular, if M is Buchsbaum and $x_i \in \mathfrak{m}^2$ for some i, then $\underline{x}$ has the monomial property. If M is generalized Cohen–Macaulay and $x_i \in \mathfrak{m}^{I(M)+1}$ for some i, then $\underline{x}$ has the monomial property, where*

$$I(M) = \sum_{1=0}^{d-1} \binom{d-1}{i} \ell\left(H_\mathfrak{m}^i(M)\right).$$

Proof. Note that $\ell(M/Q_M(\underline{x})) = \ell(\widehat{M}/Q_{\widehat{M}}(\underline{x}))$. Therefore we can assume that R is complete and $M = \widehat{M},\ N = \widehat{N}$. Let $\overline{M} = M/N$. Then for any system of parameters $\underline{x}$ of M, we have a surjection

$$f : M/Q(\underline{x}; M) \to \overline{M}/Q\left(\underline{x}; \overline{M}\right)$$

defined by $f(m + Q(\underline{x}; M)) = \overline{m} + Q(\underline{x}; \overline{M})$. Therefore, by Theorem 3.1, it is enough to prove for the case where M is generalized Cohen–Macaulay. For all $t \gg 0$, we have by [T, 3.5] that

$$\begin{aligned}\left(x_1^{t+1},\dots,x_d^{t+1}\right)M:_M x_1^t\cdots x_d^t &= (x_1,\dots,x_d)M\\ &\quad+\sum_{i=1}^{d}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)M:_M\mathfrak{q}.\end{aligned}$$

We claim that $(x_1,\dots,x_d)M:\mathfrak{q}\neq M$. In fact, set $M'=M/(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)M$. Then $\dim M'=1$. Suppose in contradiction that $(x_1,\dots,x_d)M:\mathfrak{q}=M$. Then $\mathfrak{q}M\subseteq (x_1,\dots,x_d)M$. It implies that $\mathfrak{q}M'\subseteq x_iM'\subseteq \mathfrak{q}\mathfrak{m}M'$. Therefore $\mathfrak{m}\mathfrak{q}M'=\mathfrak{q}M'$ and hence $\mathfrak{q}M'=0$ by Nakayama Lemma. So, $\dim M'\leqslant 0$, a contradiction. It follows by the claim that $\ell(M/Q_M(\underline{x}))\neq 0$ and hence $\underline{x}$ has the monomial property. The remaining statements follows from the well-known facts that every system of parameters of a Buchsbaum module (respectively of a generalized Cohen–Macaulay module) is a weak $\mathfrak{m}$-sequence (respectively a weak $\mathfrak{m}^{I(M)}$-sequence). □

Remark 3.6. To get the monomial property for modules, in general, we can not find a power less strictly than that given in Corollary 3.5. In fact, let $S=k[x,y]$ be the polynomial ring of two variables over a field k. Let $\mathfrak{m}=(x,y)S$, $R=S_{\mathfrak{m}}$ and $M=(x,y)R$. Then we have $H^0_{\mathfrak{m}}(M)=0$, $H^1_{\mathfrak{m}}(M)\cong k$. Therefore $I(M)=1$ and hence M is Buchbaum. It follows that M is pseudo generalized CM and every system of parameters of M is a weak $\mathfrak{m}$-sequence. However, the system of parameters (x,y) of M does not have the monomial property.

It is natural to ask that whether the pseudo Cohen–Macaulayness and pseudo generalized Cohen–Macaulayness are preserved by localization? The following result gives a partial answer to this question.

Proposition 3.7. *Suppose that M is quasi-unmixed* (*i.e. $\widehat{M}$ is equidimensional*). *Then the following statements are true.*

(i) *If M is pseudo CM then $M_{\mathfrak{q}}$ is pseudo CM for all $\mathfrak{q}\in\operatorname{Supp}M$.*
(ii) *If M is pseudo generalized CM then $M_{\mathfrak{q}}$ is pseudo CM for all $\mathfrak{q}\in\operatorname{Supp}M\setminus\{m\}$.*

Proof. It is clear that (i) follows from (ii). So we need only to prove (ii). Let $\widehat{N}$ be the largest submodule of $\widehat{M}$ of dimension at most $d-1$. Let $\hat{\mathfrak{q}}\in\operatorname{Supp}\widehat{M}\setminus\{\mathfrak{m}\widehat{R}\}$. Because M is pseudo generalized CM, so is $\widehat{M}$. So, we get by Theorem 3.1(ii) that $\widehat{M}/\widehat{N}$ is generalized Cohen–Macaulay. Therefore, $\widehat{M}_{\hat{\mathfrak{q}}}/\widehat{N}_{\hat{\mathfrak{q}}}$ is Cohen–Macaulay. Since $\widehat{M}$ is equidimensional, any minimal prime ideal of $\operatorname{Ass}_{\widehat{R}}\widehat{M}$ does not belong to $\operatorname{Ass}_{\widehat{R}}\widehat{N}$. Therefore $\dim\widehat{N}_{\hat{\mathfrak{q}}}<\dim\widehat{M}_{\hat{\mathfrak{q}}}$. Since $\widehat{M}_{\hat{\mathfrak{q}}}/\widehat{N}_{\hat{\mathfrak{q}}}$ is also Cohen–Macaulay, $\widehat{N}_{\hat{\mathfrak{q}}}$ is the largest of $\widehat{M}_{\hat{\mathfrak{q}}}$ of dimension at most $\dim\widehat{M}_{\hat{\mathfrak{q}}}-1$. So, $\widehat{M}_{\hat{\mathfrak{q}}}$ is pseudo CM by Theorem 3.1(i). Now let $\mathfrak{q}\in\operatorname{Supp}M\setminus\{\mathfrak{m}\}$ and $\hat{\mathfrak{q}}$ an element of $\operatorname{Ass}(\widehat{R}/\mathfrak{q}\widehat{R})$ such that $\dim\widehat{R}/\hat{\mathfrak{q}}=\dim R/\mathfrak{q}$. Let $f:R_{\mathfrak{q}}\to\widehat{R}_{\hat{\mathfrak{q}}}$ be the natural homomorphism. Since f is faithfully flat and $\dim(M_{\mathfrak{q}})=\dim(\widehat{M}_{\hat{\mathfrak{q}}})$, we can check that $pf(M_{\mathfrak{q}})=pf(\widehat{M}_{\hat{\mathfrak{q}}})$. Thus, $M_{\mathfrak{q}}$ is pseudo CM. □

Corollary 3.7 is not true, even in the case that R is a complete ring, when M is not equidimensional.

Example 3.8. Let $k \geqslant 1$ be an integer. Then there exists a pseudo CM module M and a prime ideal $\mathfrak{p} \in \operatorname{Supp} M$ such that $pf(M_\mathfrak{p}) = k$. In this case, $M_\mathfrak{p}$ is neither pseudo CM nor pseudo generalized CM.

Proof. First we assume that there exist finitely generated R-modules A, B with the following properties:

(i) A is Cohen–Macaulay.
(ii) B is of dimension at most $\dim A - 1$.
(iii) There exists $\mathfrak{p} \in \operatorname{Supp} B$ and $\mathfrak{p} \notin \operatorname{Supp} A$ such that $pf(B_\mathfrak{p}) = k$.

Then we set $M = A \oplus B$. It follows by Lemma 2.4(i) that M is pseudo CM. Since $\mathfrak{p} \in \operatorname{Supp} B$, $\mathfrak{p} \in \operatorname{Supp} M$. Since $\mathfrak{p} \notin \operatorname{Supp} A$, we have $A_\mathfrak{p} = 0$ and hence $M_\mathfrak{p} = B_\mathfrak{p}$. Therefore $pf(M_\mathfrak{p}) = k > 0$. Now we show the existence of A and B as above. Let $d \geqslant k+2$ be an integer and K a field. Let R be the formal power series ring $K[\![x_1, \ldots, x_d, y, z, t]\!]$ of $d + 3$ variables over K. Let $A = R/yR$. Then A is Cohen–Macaulay of dimension $d + 2$. Let $C = R/(z, t)R$. Then C is Cohen–Macaulay of dimension $d + 1$. Let $B = (x_1, \ldots, x_{d-k})C$. Then B is of dimension $d + 1$. Let $\mathfrak{p} = (x_1, \ldots, x_d, z, t)R$. Then $\mathfrak{p} \in \operatorname{Supp} B$ and $\mathfrak{p} \notin \operatorname{Supp} A$. We will prove that $pf(B_\mathfrak{p}) = k$. Since C is Cohen–Macaulay and $\operatorname{ht}(\mathfrak{p}/(z, t)R) = d$, $C_\mathfrak{p}$ is Cohen–Macaulay of dimension d. It is clear that $B_\mathfrak{p} = (x_1, \ldots, x_{d-k})C_\mathfrak{p}$. Therefore $B_\mathfrak{p}$ is of dimension d and $C_\mathfrak{p}/B_\mathfrak{p}$ is Cohen–Macaulay of dimension k. From the exact sequence of $R_\mathfrak{p}$-modules

$$0 \to B_\mathfrak{p} \to C_\mathfrak{p} \to C_\mathfrak{p}/B_\mathfrak{p} \to 0,$$

we have

$$H^i_{\mathfrak{p}R_\mathfrak{p}}(B_\mathfrak{p}) = \begin{cases} 0, & \text{if } i \neq k+1,\ i \neq d; \\ H^k_{\mathfrak{p}R_\mathfrak{p}}(C_\mathfrak{p}/B_\mathfrak{p}), & \text{if } i = k+1; \\ H^d_{\mathfrak{p}R_\mathfrak{p}}(C_\mathfrak{p}), & \text{if } i = d. \end{cases}$$

Therefore $\operatorname{depth}(B_\mathfrak{p}) = k + 1$. Moreover, we have by [C1, 1.1] that

$$p(B_\mathfrak{p}) = \max_{i=0,\ldots,d-1} \bigl\{\dim\bigl(R_\mathfrak{p}/\operatorname{Ann}_{R_\mathfrak{p}}\bigl(H^i_{\mathfrak{p}R_\mathfrak{p}}(B_\mathfrak{p})\bigr)\bigr)\bigr\} = k.$$

So, $\operatorname{depth}(B_\mathfrak{p}) > p(B_\mathfrak{p})$. Thus $pf(B_\mathfrak{p}) = k$ by [CM, 3.5]. $\square$

4. Sequentially Cohen–Macaulay and sequentially generalized Cohen–Macaulay modules

The concept of *sequentially Cohen–Macaulay module* was introduced by Stanley [St, p. 87] for graded modules (see also [HS]). Here we define this notion for the local case.

Definition 4.1. (i) A filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ of submodules of M is called the *dimension filtration* of M if M_{i-1} is the largest submodule of M_i which has dimension strictly less than $\dim M_i$ for all $i = 1, \ldots, t$.

(ii) A filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_t = M$ of submodules of M is said to be a *Cohen–Macaulay filtration* if

(a) Each quotient N_i/N_{i-1} is Cohen–Macaulay.
(b) $\dim N_1/N_0 < \dim N_2/N_1 < \cdots < \dim N_t/N_{t-1}$.

Definition 4.2. We say that M is a *sequentially Cohen–Macaulay module* (sequentially CM module for short) if there exists a Cohen–Macaulay filtration of M.

Similarly, we introduce the following notion.

Definition 4.3. (i) A filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_t = M$ of submodules of M is said to be a *generalized Cohen–Macaulay filtration* if

(a) Each quotient N_i/N_{i-1} is generalized Cohen–Macaulay.
(b) $\dim N_1/N_0 < \dim N_2/N_1 < \cdots < \dim N_t/N_{t-1}$.

(ii) We say that M is a *sequentially generalized Cohen–Macaulay module* (sequentially generalized CM module for short) if there exists a generalized Cohen–Macaulay filtration of M.

Lemma 4.4. *The following statements are true.*

(i) *The dimension filtration always exists and it is unique. Moreover, let $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ be a dimension filtration of M with $\dim M_i = d_i$. Then we have*

$$M_i = \bigcap_{\dim R/\mathfrak{p}_j > d_i} N_j,$$

for all $i = 1, \ldots, t-1$, where $0 = \bigcap_{j=1}^{n} N_j$ is a reduced primary decomposition of 0 in M with N_j is $\mathfrak{p}_j$-primary for $j = 1, \ldots, n$.

(ii) *Suppose that M has a Cohen–Macaulay filtration. Then it is unique and in this case, it is exactly the dimension filtration of M.*

(iii) *Suppose that M has a generalized Cohen–Macaulay filtration. Then it is unique up to $\mathfrak{m}$-primary components, i.e. if $0 = M_0 \subset M_1 \subset \cdots \subset M_{t'} = M$ is the dimension filtration of M and $0 = N_0 \subset N_1 \subset \cdots \subset N_t = M$ is a generalized Cohen–Macaulay* *filtration then $t = t'$ and $\ell(M_i/N_i) < \infty$ for all $i = 1, \dots, t-1$. Therefore in this case the dimension filtration is also a generalized Cohen–Macaulay filtration.*

Proof. (i) The unique existence of the dimension filtration follows from the Noetherian property of M. Then we get the formula by [Sch2, 2.2].

(ii) Let $0 = M_0 \subset M_1 \subset \cdots \subset M_{t'} = M$ be the dimension filtration of M and $0 = N_0 \subset N_1 \subset \cdots N_t = M$ a Cohen–Macaulay filtration of M. Since

$$\dim N_1/N_0 < \dim N_2/N_1 < \cdots < \dim N_t/N_{t-1},$$

we have $\dim N_{i-1} < \dim N_i$, for all $i = 1, \dots, t$. Therefore $M_{t'-1} \supseteq N_{t-1}$. Since M/N_{t-1} is Cohen–Macaulay, every submodule of M/N_{t-1} is zero or is of dimension d. Thus, since $\dim(M_{t'-1}/N_{t-1}) < d$, $M_{t'-1} = N_{t-1}$. Similarly, $M_{t'-2} = N_{t-2}, M_{t'-3} = N_{t-3}, \dots$. Therefore $t = t'$ and $M_i = N_i$ for all $i = 0, 1, \dots, t$.

(iii) Since M/N_{t-1} is generalized Cohen–Macaulay, every submodule of M/N_{t-1} is either of dimension d or of finite length. Thus, since $\dim(M_{t'-1}/N_{t-1}) < d$, we have $\ell(M_{t'-1}/M_{t-1}) < \infty$. Therefore, from the exact sequence

$$0 \to N_{t-1}/N_{t-2} \to M_{t'-1}/N_{t-2} \to M_{t'-1}/N_{t-1} \to 0$$

and the notice that N_{t-1}/N_{t-2} is generalized Cohen–Macaulay, we obtain that $M_{t'-1}/N_{t-2}$ is also generalized Cohen–Macaulay. Therefore every submodule of $M_{t'-1}/N_{t-2}$ is either of dimension $\dim M_{t'-1}$ or of finite length. Thus, since $\dim(M_{t'-2}/N_{t-2}) < \dim M_{t'-1}$, $\ell(M_{t'-2}/N_{t-2}) < \infty$. Continue this process, after t steps we get $t' = t$, $\ell(M_i/N_i) < \infty$ and M_i/N_{i-1} is generalized Cohen–Macaulay for all $i = 1, \dots, t$. Now, for all $i = 1, \dots, t$, from the exact sequence

$$0 \to M_{i-1}/N_{i-1} \to M_i/N_{i-1} \to M_i/M_{i-1} \to 0$$

with the notice that M_i/N_{i-1} is generalized Cohen–Macaulay, it follows that M_i/M_{i-1} is a generalized Cohen–Macaulay module. So $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ is a generalized Cohen–Macaulay filtration as required. □

The simple examples of sequentially CM modules (respectively sequentially generalized CM modules) are Cohen–Macaulay modules (respectively generalized Cohen–Macaulay modules). Especially, it follows by [G, 1.1, (1) $\Leftrightarrow$ (4)] that any approximately Cohen–Macaulay ring is sequentially CM. The following lemma produces many examples of sequentially CM modules and sequentially generalized CM modules.

Proposition 4.5. *The following statements are true.*

(i) *A direct sum of finitely many sequentially CM modules is sequentially CM.*
(ii) *A direct sum of finitely many sequentially generalized CM modules is sequentially generalized CM.*

Proof. (i) By induction, it is enough to prove for a direct sum of two sequentially CM modules. Let $M = M' \oplus M''$, where M' and M'' are sequentially CM modules. Let $\dim M = d$. We prove by induction on d that M is sequentially CM. If $d \leqslant 1$, it is trivial.

Let $d > 1$. Denote by N, N', and N'' respectively are the largest submodule of M, M', and M'' which has dimension strictly less than d. Then M'/N' and M''/N'' are zero or Cohen–Macaulay. We first claim that $N = N' \oplus N''$. In fact, since $\dim N' \oplus N'' < d$, we have $N \supseteq N' \oplus N''$. Let $a \in N$. Then $a = b + c$, where $b \in M'$ and $c \in M''$. If $\dim Rb = d$ then there exists $\mathfrak{p} \in \operatorname{Ass} M'$ such that $\dim R/\mathfrak{p} = d$ and $\mathfrak{p} = \operatorname{Ann}(rb)$ for some $r \in R$. Therefore $\mathfrak{p} \supseteq \operatorname{Ann}(ra)$ and hence $\dim Ra \geqslant d$. It gives a contradiction because $a \in N$. Thus $\dim Rb < d$. Similarly, $\dim Rc < d$. Therefore $Rb \oplus Rc \subseteq N' \oplus N''$. It follows that $a \in N' \oplus N''$ and hence $N = N' \oplus N''$. The claim is proved.

Next, we prove M/N is Cohen–Macaulay. For a system of parameters $\underline{x}$ of M, since $\dim N < d$, $\dim N' < d$, $\dim N'' < d$, we have

$$e(\underline{x}; M/N) = e(\underline{x}; M) = e(\underline{x}; M') + e(\underline{x}; M'') = e(\underline{x}; M'/N') + e(\underline{x}; M''/N'').$$

We have the exact sequence

$$0 \to \operatorname{Ker}(f) \to (M' \oplus M'')\big/\big((\underline{x}M' \oplus \underline{x}M'') + N\big) \xrightarrow{f} (M''/N'')/\underline{x}(M''/N'') \to 0,$$

where $f(\overline{b+c}) = \overline{c + N''}$, for all $b \in M',\ c \in M''$. Therefore

$$\ell\big((M' \oplus M'')\big/\big((\underline{x}M' \oplus \underline{x}M'') + N\big)\big) \leqslant \ell\big((M''/N'')/\underline{x}(M''/N'')\big) + \ell(\operatorname{Ker} f).$$

It is clear that $\operatorname{Ker} f = (M' \oplus (\underline{x}M'' + N''))/((\underline{x}M' \oplus \underline{x}M'') + N)$. Moreover, we have a surjection

$$p : (M'/N')/\underline{x}(M'/N') \to \big(M' \oplus (\underline{x}M'' + N'')\big)\big/\big((\underline{x}M' \oplus \underline{x}M'') + N\big)$$

which is defined by $p(\overline{b + N'}) = \overline{b + 0}$, for all $b \in M'$. Therefore we have

$$\begin{aligned} \ell\big((M/N)/\underline{x}(M/N)\big) &= \ell\big((M' \oplus M'')\big/\big((\underline{x}M' \oplus \underline{x}M'') + N\big)\big) \\ &\leqslant \ell\big((M'/N')/\underline{x}(M'/N')\big) + \ell\big((M''/N'')/\underline{x}(M''/N'')\big) \\ &= e(\underline{x}; M'/N') + e(\underline{x}; M''/N'') \\ &= e(\underline{x}; M/N). \end{aligned}$$

Thus M/N is Cohen–Macaulay. Since $N = N' \oplus N''$, N' and N'' are also sequentially CM modules and $\dim N < d$, we can apply induction hypothesis to N, and we get that N is sequentially CM. Therefore M is sequentially CM.

(ii) follows similarly to the proof of (i). □

Lemma 4.6. *Let M be a sequentially generalized CM module. Then* $\operatorname{Supp} M$ *is a catenary subset of* $\operatorname{Spec} R$.

Proof. It is clear that

$$\operatorname{Supp} M = \bigcup_{i=1}^{t} \operatorname{Supp} M_i/M_{i-1}.$$

Since M_i/M_{i-1} is generalized Cohen–Macaulay, it follows by [CST] that $\operatorname{Supp} M_i/M_{i-1}$ is catenary for all $i = 1, \ldots, t$. Therefore so is $\operatorname{Supp} M$. $\square$

Proposition 4.7. *The following statements are true.*

(i) *If M is sequentially CM then so is $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp} M$.*
(ii) *If M is sequentially generalized CM then for all $\mathfrak{p} \in \operatorname{Supp} M \backslash \{m\}$, $M_{\mathfrak{p}}$ is sequentially CM.*

Proof. It is clear that (i) follows immediately by (ii). Therefore it is enough to prove (ii). Let $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ be the dimension filtration of M. Then, it follows from Lemma 4.4(iii) that M_i/M_{i-1} is generalized Cohen–Macaulay for all $i = 1, \ldots, t$. Let $\mathfrak{p} \in \operatorname{Supp} M \backslash \{\mathfrak{m}\}$. We claim that $(M_{t-1})_{\mathfrak{p}} = M_{\mathfrak{p}}$ or $\dim(M_{t-1})_{\mathfrak{p}} < \dim M_{\mathfrak{p}}$. In fact, suppose that $(M_{t-1})_{\mathfrak{p}} \neq M_{\mathfrak{p}}$. Then $\mathfrak{p} \in \operatorname{Supp} M/M_{t-1}$. Therefore there exists $\mathfrak{q} \in \operatorname{Ass} M$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $\dim R/\mathfrak{q} = d$. Since M/M_{t-1} is generalized Cohen–Macaulay, it follows by Lemma 4.6 that

$$\begin{aligned} \dim(M_{t-1})_{\mathfrak{p}} &\leqslant \dim M_{t-1} - \dim R/\mathfrak{p} < d - \dim R/\mathfrak{p} \\ &= \dim R/\mathfrak{q} - \dim R/\mathfrak{p} = ht(\mathfrak{p}/\mathfrak{q}) \leqslant \dim M_{\mathfrak{p}}. \end{aligned}$$

Continue this process we obtain $(M_{i-1})_{\mathfrak{p}} = (M_i)_{\mathfrak{p}}$ or $\dim(M_{i-1})_{\mathfrak{p}} < \dim(M_i)_{\mathfrak{p}}$ for all $i = 1, \ldots, t$. Thus, from the family $\{(M_i)_{\mathfrak{p}}\}_{i=0,1,\ldots,t}$ of submodules of $M_{\mathfrak{p}}$, we can choose a Cohen–Macaulay filtration of submodules of $M_{\mathfrak{p}}$

$$0 = (M_{i_0})_{\mathfrak{p}} \subset (M_{i_1})_{\mathfrak{p}} \subset \cdots \subset (M_{i_{t_1}})_{\mathfrak{p}} = M_{\mathfrak{p}}.$$

Thus $M_{\mathfrak{p}}$ is sequentially CM. $\square$

5. Cohomological characterizations of sequentially CM and sequentially generalized CM modules

Suppose that R possesses a dualizing complex. Then there is a bounded complex $D_R^{\bullet}$ of injective R-modules D_R^i whose cohomology modules $H^i(D_R^{\bullet})$, $i \in \mathbb{Z}$, are finitely generated R-modules. For a finitely generated R-module M of $\dim M = d$, the homology module

$$K^i(M) := H^{-i}\big(\operatorname{Hom}\big(M, D_R^{\bullet}\big)\big)$$

is also a finitely generated R-module, for all $i = 0, 1, \dots, d$. Note that the module $K^d(M)$ is just the canonical module of M. Following [Sch1], for $i = 0, 1, \dots, d-1$, the module $K^i(M)$ is called *ith module of deficiency* of M. Moreover, by the local duality (see [Sch1, 1.1]) there are following isomorphisms:

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Hom}\bigl(K^i(M), E\bigr), \quad \forall i,$$

where E is the injective hull of $R/\mathfrak{m}$. The two following theorems give a cohomological characterization of sequentially CM and sequentially generalized CM modules.

Theorem 5.1. *Let $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ be the dimension filtration of M and $d_i = \dim M_i$ for $i = 1, \dots, t$. Suppose that R possesses a dualizing complex.*

(a) *The following statements are equivalent*:
  (i) *M is sequentially CM.*
  (ii) *M_i is pseudo CM for all $i = 1, \dots, t$.*
  (iii) *For all $j = 0, 1, \dots, d$ the modules $K^j(M)$ are either zero or Cohen–Macaulay of dimension j.*
  (iv) *For all $j = 0, 1, \dots, d-1$ the modules $K^j(M)$ are either zero or Cohen–Macaulay of dimension j.*

(b) *Suppose that M satisfies the equivalent conditions above. Then*

$$d_{i-1} = \dim M_{i-1} = p(M_i)$$

for all $i = 1, \dots, t$, where $p(M_i)$ is the polynomial type of M_i.

Proof. (a): (i) $\Leftrightarrow$ (ii). Let M be sequentially CM. Then we get by Lemma 4.4(ii) that M_i/M_{i-1} is Cohen–Macaulay for all $i = 1, \dots, t$. Therefore, by Lemma 2.4(i), M_i is pseudo CM for all $i = 1, \dots, t$. The converse follows immediately by Theorem 3.1(i).

(i) $\Rightarrow$ (iii). Let M be sequentially CM. Then M_i/M_{i-1} is Cohen–Macaulay for all $i = 1, \dots, t$. It can be derived from the exact sequence

$$0 \to M_{t-1} \to M \to M/M_{t-1} \to 0$$

that $K^d(M) \cong K^d(M/M_{t-1})$, $K^j(M) = 0$ for all $j = d_{t-1} + 1, \dots, d-1$, and $K^j(M) \cong K^j(M_{t-1})$ for all $j = 0, \dots, d_{t-1}$. Since M/M_{t-1} is Cohen–Macaulay, it follows that $K^d(M)$ is Cohen–Macaulay of dimension $d = d_t$. Similarly, by applying to the exact sequence

$$0 \to M_{t-2} \to M_{t-1}/M_{t-2} \to 0$$

with notice that M_{t-1}/M_{t-2} is Cohen–Macaulay, we have $K^j(M) \cong K^j(M_{t-2})$ for all $j = 0, \dots, d_{t-2}$, $K^j(M) = 0$ for all $j = d_{t-2} + 1, \dots, d_{t-1} - 1$, and $K^{d_{t-1}}(M)$ is Cohen–Macaulay of dimension d_{t-1}. Continuing this process, we get the result.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i). We prove by induction on d that M_i/M_{i-1} is Cohen–Macaulay for all $i = 1, \ldots, t$. If $d = 1$, it is trivial. Let $d > 1$. It follows by [CK, 1.1] that M is pseudo CM. So M/M_{t-1} is Cohen–Macaulay by Theorem 3.1(i). Therefore we get from the exact sequence

$$0 \to M_{t-1} \to M \to M/M_{t-1} \to 0$$

that $K^i(M) \cong K^i(M_{t-1})$ for all $i = 1, \ldots, \dim M_{t-1}$. It follows that M_{t-1} satisfies the hypothesis of (iv). Since $\dim M_{t-1} < d$, by applying the inductive assumption to M_{t-1}, we obtain that M_i/M_{i-1} is Cohen–Macaulay for all $i = 1, \ldots, t-1$. Therefore M is sequentially CM.

(b) Set $\mathfrak{a}_j = \operatorname{Ann}(H^j_{\mathfrak{m}}(M_i))$ for $i = 1, \ldots, t$ and $j = 0, \ldots, d_i - 1$. We have by the proof of (a), (i) $\Rightarrow$ (iii), that $K^j(M_i) = 0$ for all $j = d_{i-1} + 1, \ldots, d_i - 1$ and $K^{d_{i-1}}(M_i)$ is Cohen–Macaulay of dimension d_{i-1}. Therefore, by [C1, 1.1] and [Sch1, 2.2.4], we have

$$p(M_i) = \max_{j=0,\ldots,d_i-1} \dim R/\mathfrak{a}_j = \max_{j=0,\ldots,d_{i-1}} \dim K^j(M_i) = d_{i-1}$$

for all $i = 1, \ldots, t$. $\square$

It should be noted that the equivalences of statements (i), (iii), and (iv) of Theorem 5.1 has been shown by P. Schenzel [Sch2, Theorem 5.5] by using spectral sequences and (i) $\Leftrightarrow$ (iii) have been proved by Herzog–Sbarra [HS, Theorem 1.4] for a standard graded Cohen–Macaulay k-algebra R.

The following consequence gives us the structure of local cohomology modules of a sequentially CM module. Note that in this corollary R does need to possess a dualizing complex.

Corollary 5.2. *Let M be a sequentially CM module and $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ the dimension filtration of M. Set $d_i = \dim M_i$ and $\mathfrak{a}_j = \operatorname{Ann}(H^j_{\mathfrak{m}}(M))$ for $j = 0, 1, \ldots, d$. Then $H^j_{\mathfrak{m}}(M) = 0$ if and only if $j \notin \{d_1, \ldots, d_t\}$ and $\dim R/\mathfrak{a}_j = j$ for all $j \in \{d_1, \ldots, d_t\}$.*

Proof. Denote by $\widehat{N}$ the $\mathfrak{m}$-adic completion of a module N. Then $0 = \widehat{M}_0 \subset \widehat{M}_1 \subset \cdots \subset \widehat{M}_t = \widehat{M}$ is clearly a Cohen–Macaulay filtration of $\widehat{M}$. It follows by applying Theorem 5.1(b) for $\widehat{M}$ that $H^j_{\mathfrak{m}}(\widehat{M}) = 0$ and therefore $H^j_{\mathfrak{m}}(M) = 0$ if $j \notin \{d_1, \ldots, d_t\}$. On the other hand, by the proof of (i) $\Rightarrow$ (iii) in Theorem 5.1, we get for all $j = 1, \ldots, d$ that $H^{d_j}_{\mathfrak{m}}(M) \cong H^{d_j}_{\mathfrak{m}}(M_j)$. Then it implies from the basic facts of local cohomology theory that $\dim R/\mathfrak{a}_{d_j} = d_j$, as required. $\square$

To give a characterization of sequentially generalized CM modules, we need some facts of the theory of secondary representation of Artinian modules from [M,SH]: Any Artinian R-module A has a *minimal secondary representation* $A = A_1 + \cdots + A_n$ of $\mathfrak{p}_i$-secondary submodules A_i. The set $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ is independent of the choice of minimal representation of A and is denoted by $\operatorname{Att}_R A$. From now on, for any positive integer j we set $(\operatorname{Att} A)_j = \{\mathfrak{p} \in \operatorname{Att} A\colon \dim R/\mathfrak{p} = j\}$. Note that $\ell(A/\mathfrak{m}^n A)$ is finite and independent of n when n large. Therefore we denote this length by $Rl(A)$ for n large. It is clear that if $x \in \mathfrak{m}$ and $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att} A \setminus \{\mathfrak{m}\}$ then $\ell(A/x^n A) = Rl(A)$ for n large.

Theorem 5.3. *Let $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ be the dimension filtration of M and $d_i = \dim M_i$ for $i = 1, \ldots, t$. Suppose that R possesses a dualizing complex. Then*

(a) *The following statements are equivalent*:
  (i) *M is sequentially generalized CM.*
  (ii) *M_i is pseudo generalized CM for all $i = 1, \ldots, t$.*
  (iii) *For all $j = 1, \ldots, d$, the modules $K^j(M)$ are either of finite length or generalized Cohen–Macaulay of dimension j.*
  (iv) *For all $j = 1, \ldots, d-1$, the modules $K^j(M)$ are either of finite length or generalized Cohen–Macaulay of dimension j.*

(b) *Suppose that M satisfies the equivalent conditions above. Then*

$$d_{i-1} = \dim M_{i-1} = p(M_i),$$

where $p(M_i)$ is the polynomial type of M_i for all $i = 1, \ldots, t$.

Proof. (a): (i) $\Leftrightarrow$ (ii). Assume that M is sequentially generalized CM. Then M_i/M_{i-1} is generalized Cohen–Macaulay by Lemma 4.4(iii). Thus, M_i is pseudo generalized CM for all $i = 1, \ldots, t$ by Lemma 2.4(ii). The converse follows immediately by Theorem 3.1(ii).

(i) $\Rightarrow$ (iii). Suppose that M is sequentially generalized CM. Then M_i/M_{i-1} is generalized Cohen–Macaulay for all $i = 1, \ldots, t$. We claim that $K^{d_i}(M_i/M_{i-1})$ is generalized Cohen–Macaulay for all $i = 1, \ldots, t$. In fact, let $\mathfrak{p} \in \operatorname{Supp} K^{d_i}(M_i/M_{i-1}) \setminus \{\mathfrak{m}\}$. Then $\mathfrak{p} \in \operatorname{Supp}(M_i/M_{i-1}) \setminus \{\mathfrak{m}\}$. Therefore we have by [Sch1, 2.2.3] and [CK] that $(K^{d_i}(M_i/M_{i-1}))_{\mathfrak{p}}$ is Cohen–Macaulay and therefore the claim follows by [CST]. Similarly to the proof of Theorem 5.1, (i) $\Rightarrow$ (iii), and by the claim, it follows that $K^j(M)$ is generalized Cohen–Macaulay of dimension j for all $j = d_1, \ldots, d_t$, and $\ell(K^j(M)) < \infty$ for all $j \notin \{d_1, \ldots, d_t\}$.

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i). Let $\mathfrak{a}(M) = \mathfrak{a}_0(M)\mathfrak{a}_1(M)\ldots\mathfrak{a}_{d-1}(M)$, where $\mathfrak{a}_i(M) = \operatorname{Ann} H^i_{\mathfrak{m}}(M)$, $i = 0, \ldots, d-1$. Then there exists by [C1] a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M such that $x_d \in \mathfrak{a}(M)$ and $x_i \in \mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M)$, for all $i = 1, \ldots, d-1$. A such system of parameters is called a *p-standard system of parameters.* First of all we show the following claim.

Claim. *Let $\underline{x} = (x_1, \ldots, x_d)$ be a p-standard system of parameters of M. Then $J_{M,\underline{x}}(\underline{n})$ is bounded above by a constant for all $\underline{n} = (n_1, \ldots, n_d)$.*

Proof of the claim. Let $\underline{n} = (n_1, \ldots, x_d)$ be a set of positive integers. For short we put $M_j = M/(x_1^{n_1}, \ldots, x_j^{n_j})M$, $\underline{x}_j = (x_{j+1}, \ldots, x_d)$, and $\underline{n}_j = (n_{j+1}, \ldots, n_d)$ for all $j = 1, \ldots, d-1$. It follows by [C1, 3.4] and [CM, 2.5] that $\ell(0 :_{M_{j-1}} x_j^{n_j}) < \infty$ for all j.

Therefore $x_j \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M_{j-1} \setminus \{\mathfrak{m}\}$. Hence $x_j \notin \mathfrak{p}$ for all $\mathfrak{p} \in (\operatorname{Att}(H^i_{\mathfrak{m}}(M_{j-1})))_i$. So by the Matlis duality (see [BS, 10.2.20]),

$$\big(\operatorname{Ass}\big(0 :_{K^i(M_{j-1})} x_j^{n_j}\big)\big)_i = \big(\operatorname{Att}\big(H^i_{\mathfrak{m}}(M_{j-1})\big)\big/x_j^{n_j} H^i_{\mathfrak{m}}(M_{j-1})\big)_i = \emptyset.$$

Hence

$$\dim\big(0 :_{K^i(M_{j-1})} x_j^{n_j}\big) \leqslant i - 1 \tag{1}$$

for all $i = 1, \ldots, d - j + 1$. First, we prove by induction on j that $K^i(M_j)$ is either of finite length or generalized Cohen–Macaulay of dimension i, for all $i = 1, \ldots, d - j$. The case where $j = 0$ follows by the hypothesis. Let $j > 0$. From the exact sequences

$$0 \to 0 :_{M_{j-1}} x_j^{n_j} \to M_{j-1} \to M_{j-1}/0 :_{M_{j-1}} x_j^{n_j} \to 0,$$
$$0 \to M_{j-1}/0 :_{M_{j-1}} x_j^{n_j} \xrightarrow{x_j^{n_j}} M_{j-1} \to M_j \to 0$$

with notice that $\ell(0 :_{M_{j-1}} x_j^{n_j}) < \infty$, we get by the local duality the following exact sequence:

$$0 \to K^{i+1}(M_{j-1})\big/x_j^{n_j} K^{i+1}(M_{j-1}) \to K^i(M_j) \to 0 :_{K^i(M_{j-1})} x_j^{n_j} \to 0 \tag{2}$$

for $i = 1, \ldots, d - j$. By induction hypothesis, either $\ell(K^i(M_{j-1})) < \infty$ or $K^i(M_{j-1})$ is generalized Cohen–Macaulay of dimension i. Therefore any submodule of $K^i(M_{j-1})$ is either of finite length or is of dimension i. It follows by (1) that $\ell(0 :_{K^i(M_{j-1})} x_j^{n_j}) < \infty$ and therefore by the inductive hypothesis that $K^{i+1}(M_{j-1})/x_j^{n_j} K^{i+1}(M_{j-1})$ is generalized Cohen–Macaulay. Hence $K^i(M_j)$ is either of finite length or generalized Cohen–Macaulay of dimension i for all $i = 1, \ldots, d - j$ by (2). On the other hand, since $J_{M_j,\underline{x}_j}(\underline{n}_j) = J_{M_j/H^0_{\mathfrak{m}}(M_j),\underline{x}_j}(\underline{n}_j)$, we can assume that x_{j+1} is non-zero-divisor of M_j. Therefore it can be derived by [CM, 2.1] that

$$J_{M_j,\underline{x}_j}(\underline{n}_j) \leqslant J_{M_{j+1},\underline{x}_{j+1}}(\underline{n}_{j+1}) + Rl\big(H^{d-j-1}_{\mathfrak{m}}(M_j)\big)$$

for all $j = 0, \ldots, d - 1$. Note that $J_{M_{d-1},(x_d)}(n_d) = 0$. Therefore we have

$$J_{M,\underline{x}}(\underline{n}) \leqslant \sum_{j=1}^{d-1} Rl\big(H^j_{\mathfrak{m}}(M_{d-j-1})\big).$$

Next, since $Rl(H^i_{\mathfrak{m}}(M_j)) = \ell(H^0_{\mathfrak{m}}(K^i(M_j)))$, the claim is proved if we can show that $\ell(H^k_{\mathfrak{m}}(K^i(M_j)))$ is bounded above by a constant for all $j = 1, \ldots, d - 1$, $i = 1, \ldots,$

$d-j-1$, and $k=0,\dots,i-1$. Indeed, by the exact sequence (2), we get the following exact sequence:

$$H^k_{\mathfrak{m}}\big(K^{i+1}(M_{j-1})\big/x_j^{n_j}K^{j+1}(M_{j-1})\big)\to H^k_{\mathfrak{m}}\big(K^i(M_j)\big)\to H^k_{\mathfrak{m}}\big(0:_{K^i(M_{j-1})}x_j^{n_j}\big),$$

for all $k=0,\dots,i-1$. Since $\ell(0:_{K^i(M_{j-1})}x_j^{n_j})<\infty$, $H^k_{\mathfrak{m}}(0:_{K^i(M_{j-1})}x_j^{n_j})=0$ for all $k>0$ and

$$H^0_{\mathfrak{m}}\big(0:_{K^i(M_{j-1})}x_j^{n_j}\big)=\big(0:_{K^i(M_{j-1})}x_j^{n_j}\big)\subseteq H^0_{\mathfrak{m}}\big(K^i(M_{j-1})\big).$$

Therefore we obtain

$$\ell\big(H^k_{\mathfrak{m}}\big(K^i(M_j)\big)\big)\leqslant \ell\big(H^k_{\mathfrak{m}}\big(K^{i+1}(M_{j-1})\big/x_j^{n_j}K^{i+1}(M_{j-1})\big)\big) \tag{3}$$

for $k>0$ and

$$\begin{aligned}\ell\big(H^0_{\mathfrak{m}}\big(K^i(M_j)\big)\big) &\leqslant \ell\big(H^0_{\mathfrak{m}}\big(K^{i+1}(M_{j-1})\big/x_j^{n_j}K^{i+1}(M_{j-1})\big)\big)\\ &\quad+\ell\big(H^0_{\mathfrak{m}}\big(K^i(M_{j-1})\big)\big).\end{aligned} \tag{4}$$

From the short exact sequence

$$\begin{aligned}0 &\to K^{i+1}(M_{j-1})\big/0:_{K^i(M_{j-1})}x_j^{n_j}\to K^{i+1}(M_{j-1})\\ &\to K^{i+1}(M_{j-1})\big/x_j^{n_j}K^{i+1}(M_{j-1})\to 0\end{aligned}$$

we get the following exact sequence:

$$H^k_{\mathfrak{m}}\big(K^{i+1}(M_{j-1})\big)\to H^k_{\mathfrak{m}}\big(K^{i+1}(M_{j-1})\big/x_j^{n_j}K^{i+1}(M_{j-1})\big)\to H^{k+1}_{\mathfrak{m}}\big(K^{i+1}(M_{j-1})\big)$$

for all $k=0,\dots,i-1$. Thus, with the observation that $\ell(0:_{K^i(M_{j-1})}x_j^{n_j})<\infty$ and $K^{i+1}(M_{j-1})$ is generalized Cohen–Macaulay, we have by (3), (4) that

$$\begin{aligned}\ell\big(H^k_{\mathfrak{m}}\big(K^i(M_j)\big)\big) &\leqslant \ell\big(H^k_{\mathfrak{m}}\big(K^{i+1}(M_{j-1})\big)\big)+\ell\big(H^{k+1}_{\mathfrak{m}}\big(K^{i+1}(M_{j-1})\big)\big)\\ &\quad+\ell\big(H^0_{\mathfrak{m}}\big(K^i(M_{j-1})\big)\big)\end{aligned}$$

for all $j=1,\dots,d-1$, $i=1,\dots,d-j-1$, and $k=0,\dots,i-1$. Then the result follows easily by induction on j.

Now we continue to prove Theorem 5.3. We prove by induction on d that M is sequentially generalized CM. If $d=1$, it is trivial. Suppose $d>1$. It follows by the claim and Theorem 2.1(ii) that M is pseudo generalized CM. Therefore M/M_{t-1} is generalized Cohen–Macaulay by Theorem 3.1(ii). From the exact sequence

$$0\to M_{t-1}\to M\to M/M_{t-1}\to 0$$

we get the exact sequences

$$K^j(M/M_{t-1}) \to K^j(M) \to K^j(M_{t-1}) \to K_{\mathfrak{m}}^{j-1}(M/M_{t-1})$$

for all $j = 1, \ldots, d-1$. Let N_j be the kernel of the map $K^j(M/M_{t-1}) \to K^i(M)$ and P_j be the image of the map $K^j(M_{t-1}) \to K^{j-1}(M/M_{t-1})$ in the above exact sequences. Then N_j and P_j are of finite length. Therefore from the exact sequence

$$0 \to K^j(M)/N_j \to K^j(M_{t-1}) \to P_j \to 0$$

for all $i = 1, \ldots, d-1$, we can check that M_{t-1} satisfies the hypothesis of (iv). By applying the induction assumption to M_{t-1}, the modules M_i/M_{i-1} is generalized Cohen–Macaulay for all $i = 1, \ldots, t-1$. Therefore M is sequentially generalized CM. The statement (b) follows similarly to the proof of Theorem 5.1(b). □

Analogous to Corollary 5.2, we get the following consequence about the local cohomology modules of a sequentially generalized CM module.

Corollary 5.4. *Suppose that M is a sequentially generalized CM module and $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ the dimension filtration of M. Set $d_i = \dim M_i$ and $\mathfrak{a}_j = \mathrm{Ann}(H_{\mathfrak{m}}^j(M))$ for $j = 0, 1, \ldots, d$. Then $\ell(H_{\mathfrak{m}}^j(M)) < \infty$ if and only if $j \notin \{d_1, \ldots, d_t\}$ and $\dim R/\mathfrak{a}_j = j$ for all $j \in \{d_1, \ldots, d_t\}$.*

The following example show that Theorem 5.3 is not true in general if R does not admit a dualizing complex.

Example 5.5. There exists a finitely generated R-module M such that M is not sequentially generalized CM, but the dimension filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ has properties M_i is pseudo CM for all $i = 1, \ldots, t$.

Proof. Denote by $(A, \mathfrak{m})$ the Noetherian local domain of dimension 2 constructed by D. Ferrand and M. Raynaud in [FR] for which the $\mathfrak{m}$-adic completion $\widehat{A}$ of A has an associated prime $\mathfrak{q}$ of dimension 1 (see also [N, Appendix, Example 2]). Let $R = a[\![x_1, \ldots, x_d]\!]$ be the ring of formal series of variables $x_1, \ldots, x_d$ over A. Let $M_i = R/(x_i, \ldots, x_d)R$, $i = 1, \ldots, d$. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_d \oplus R$ and $N_0 = 0$, $N_i = M_1 \oplus M_2 \oplus \cdots \oplus M_i$ for $i = 1, \ldots, d$, and $N_{d+1} = M$. Then $\dim M = d + 2$ and

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_d \subset N_{d+1} = M$$

is the dimension filtration of M. We have $N_i/N_{i-1} \cong M_i \cong A[\![x_1, \ldots, x_{i-1}]\!]$. Let (f, g) be a system of parameters of A. Then $\underline{z} = (f, g, x_1, \ldots, x_{i-1})$ is a system of parameters of M_i. Since $x_1, \ldots, x_{i-1}, f$ is a regular sequence of M_i, it follows that $J_{M_i, \underline{z}}(\underline{n}) = 0$ for all $\underline{n} = (n_f, n_g, n_1, \ldots, n_{i-1})$. Therefore M_i is pseudo CM for all $i = 1, \ldots, d+1$. However, N_i/N_{i-1} is never generalized Cohen–Macaulay. Therefore M is not sequentially generalized CM. □

Now we study the unmixedness of $p(M)$-th module of deficiency of pseudo CM and pseudo generalized CM modules.

Proposition 5.6. *Suppose that R has a dualizing complex. Let $p = p(M)$ be the polynomial type of M. Then we have*

(i) *If M is pseudo CM then $K^p(M)$ is unmixed.*
(ii) *If M is pseudo generalized CM then $K^p(M)$ is unmixed up to an $\mathfrak{m}$-primary component.*

Proof. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ be the dimension filtration of M. We prove (i). Since M is pseudo CM, it follows by Theorem 3.1(i) that M/M_{t-1} is Cohen–Macaulay. Therefore $K^i(M) \cong K^i(M_{t-1})$ for all $i < d$. Therefore we get by Lemma 2.6(i) that $K^i(M_{t-1}) = 0$ for all $i = p+1, \ldots, d-1$. Hence $\dim M_{t-1} \leqslant p$. For $i = 0, \ldots, d-1$, we get by [Sch1, 2.2.4] that $\dim K^i(M) \leqslant i$ for all $i = 0, \ldots, d-1$. Therefore, by [C1, 1.1] and Lemma 2.6(i), we have

$$p = \max_{i=0,\ldots,d-1} \dim K^i(M) = \max_{i=0,\ldots,p} \dim K^1(M_{t-1}) \leqslant p.$$

It implies that $\dim M_{t-1} = p$ and hence $K^p(M_{t-1})$ is unmixed. Thus $K^p(M)$ is unmixed.

We prove (ii). The case where $p(M) \leqslant 0$ is trivial. Assume that $p(M) > 0$. Since M is pseudo generalized CM, the module M/M_{t-1} is generalized Cohen–Macaulay by Theorem 3.1(ii). Therefore from the exact sequence $0 \to M_{t-1} \to M \to M/M_{t-1} \to 0$, we get the exact sequences

$$0 \to K^i(M)/Q_i \to K^i(M_{t-1}) \to P_i \to 0$$

for all $i < d$, where $\ell(Q_i) < \infty$ and $\ell(P_i) < \infty$. So, $\dim K^i(M) = \dim K^i(M_{t-1})$ for all $i < d$. Therefore we have by Lemma 2.6(ii) that $K^i(M_{t-1})$ has finite length for all $i = p+1, \ldots, d-1$. It follows that $\dim M_{t-1} \leqslant p$. Since $p(M) > 0$, it follows by [Sch1, 2.2.4] and [C1, 1.1] that $\dim M_{t-1} = p$. Therefore $K^p(M_{t-1})$ is unmixed, and hence $K^p(M)$ is unmixed up to an $\mathfrak{m}$-primary component. □

The following result follows immediately by the proof of Proposition 5.6.

Corollary 5.7. *Let M be pseudo generalized CM and N the largest submodule of M of dimension at most $d-1$. Then we have $p(M) = \dim N$.*

References

[AB] M. Auslander, D.A. Buchsbaum, Codimension and multiplicity, Ann. of Math. 68 (1958) 625–657.

[BS] M. Brodmann, R.Y. Sharp, Local Cohomology: An Algebraic Introduction with Geometric Applications, Cambridge Univ. Press, 1998.

[C1] N.T. Cuong, On the dimension of the non-Cohen–Macaulay locus of local ring admitting dualizing complexes, Math. Proc. Cambridge Philos. Soc. (2) 109 (1991) 479–488.

[C2] N.T. Cuong, On the least degree of polynomials bounding above the differences between lengths and multiplicities of certain systems of parameters in local rings, Nagoya Math. J. 125 (1992) 105–114.
[CK] N.T. Cuong, V.T. Khoi, Modules whose local cohomology modules have Cohen–Macaulay Matlis duals, in: D. Eisenbud (Ed.), Proc. of Hanoi Conf. on Commutative Algebra, Algebraic Geometry, and Computational Methods, Springer-Verlag, 1999, pp. 223–231.
[CM] N.T. Cuong, N.D. Minh, Lengths of generalized fractions of modules having small polynomial type, Math. Proc. Cambridge Philos. Soc. (2) 128 (2000) 269–282.
[CMN] N.T. Cuong, M. Morales, L.T. Nhan, Lengths of generalized fractions, Prépublication de l'Institut Fourier n° 539, 2001.
[CST] N.T. Cuong, P. Schenzel, N.V. Trung, Verallgemeinerte Cohen–Macaulay Moduln, Math. Nachr. 85 (1978) 57–75.
[FR] D. Ferrand, M. Raynaund, Fibres formelles d'un anneau local noetherian, Ann. Sci. École Norm. Sup. 3 (4) (1970) 295–311.
[G] S. Goto, Approximately Cohen–Macaulay rings, J. Algebra 76 (1981) 214–225.
[GK] J.-L. Garcia Roig, D. Kirby, On the Koszul homology modules for the powers of a multiplicity system, Mathematika 33 (1986) 96–101.
[H] M. Hochster, Contracted ideals from integral extensions of regular rings, Nagoya Math. J. 51 (1973) 25–43.
[HS] J. Herzog, E. Sbarra, Sequentially Cohen–Macaulay modules and local cohomology, Preprint.
[M] I.G. Macdonald, Secondary representation of modules over a commutative ring, Sympos. Math. 11 (1973) 23–43.
[N] M. Nagata, Local Rings, Interscience, New York, 1962.
[Sch1] P. Schenzel, Dualisierende Komplexe in der lokalen Algebra und Buchsbaum Ringe, in: Lecture Notes in Math., Vol. 907, Springer-Verlag, Berlin, 1982.
[Sch2] P. Schenzel, On the dimension filtration and Cohen–Macaulay filtered modules, in: Proc. of the Ferrara Meeting in honour of Mario Fiorentini, University of Antwerp, Wilrijk, Belgium, 1998, pp. 245–264.
[SH] R.Y. Sharp, M.A. Hamieh, Lengths of certain generalized fractions, J. Pure Appl. Algebra 38 (1985) 323–336.
[St] R.P. Stanley, Combinatorics and Commutative Algebra, Second edition, Birkhäuser Boston, 1996.
[SV] J. Stückrad, W. Vogel, Buchsbaum Rings and Applications, Springer-Verlag, Berlin, 1986.
[T] N.V. Trung, Toward a theory of generalized Cohen–Macaulay modules, Nagoya Math. J. 102 (1986) 1–49.